Visualizing and representing the evolution of topological features

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Abstract

Simplicial complexes are discrete representations of topological spaces that are practical for computational studies. The first three Betti-numbers (indicating the number of components, tunnels and voids), as well as the topological persistence of each such feature, is well-defined and can be efficiently computed for simplicial complexes embedded in 2D and 3D [1, 2].

We introduce a novel representation of the evolution of topological features in simplicial complexes using so-called tunnel-trees in 2D and void-trees in 3D. This new representation makes it possible to analyze topological evolution by applying tools for analysis of binary trees. Furthermore it supplies a new method for visualizing topological evolution.

Introduction

A simplicial complex, $\mathcal{K}$, is a set of simplices where any face of a simplex in $\mathcal{K}$ is also in $\mathcal{K}$ and the intersection of two simplices in $\mathcal{K}$ is either empty or a face of both simplices. Delfinado and Edelsbrunner [1] define a *filter* to be a sequence of simplices, $\sigma^1, \sigma^2, \ldots, \sigma^n$, where $\mathcal{K}_i = \{\sigma^1, \sigma^2, \ldots, \sigma^i\}$ is a simplicial complex for any choice of i (see left part of Figure 1). The filter represents the evolution of a simplicial complex and will be the focus of the methods described here. The topological features of a complex can be described using the Betti-numbers, β_d, which indicate the rank of the dth homology group. The first three Betti-numbers $(\beta_0, \beta_1, \beta_2)$ can be interpreted more intuitively as the number of components, holes, and voids respectively. A $\mathcal{O}(n\alpha(n))$-time algorithm exists to calculate the evolution of β_d as a simplicial complex is grown using a filter [1]. This method identifies each k-simplex, σ^i, as either *positive* if it creates a new k-cycle and thereby increases β_k, or *negative* if it changes a k-cycle into a k-boundary and thereby decreases β_{k-1}. For each positive k-simplex, σ^i, the negative $(k+1)$-simplex, σ^j, that is responsible for turning the k-cycle, created by σ^i, into a k-boundary can be efficiently identified [2]. The difference between the indices of such two simplices is defined to be the *persistence* of the k-cycle represented by σ^i.

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Tunnel- and void-trees

One interesting observation about tunnels in simplicial complexes embedded in 2D is that, often, when a positive 1-simplex (edge) is added to the complex, it splits one tunnel in two. If the empty space around the complex is considered a *bounding tunnel*, then every positive edge will split an existing tunnel in two. Similarly, if the entire space around a simplicial complex embedded in 3D is considered a *bounding void*, then a positive 2-simplex (triangle) always splits an existing void in two.

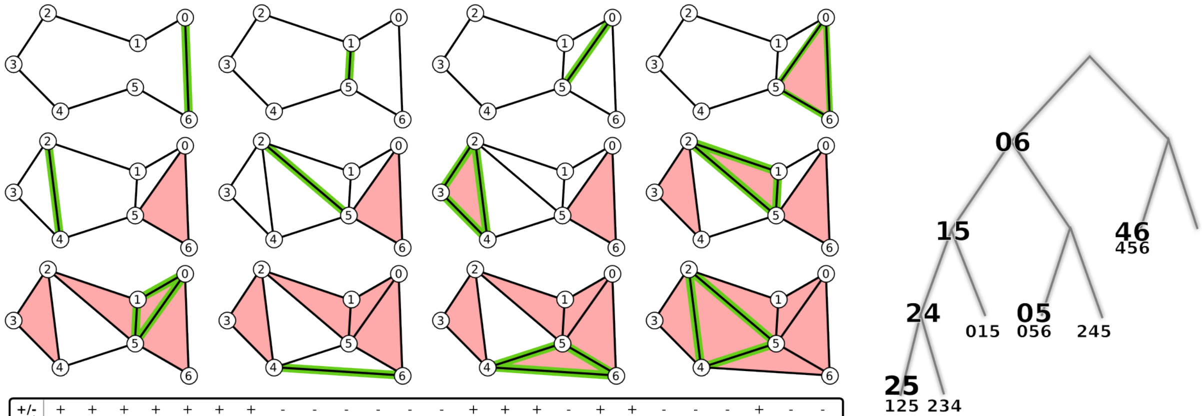
Based on this observation we define a *tunnel-tree* (or β_1-tree) of a 2D filter to be a binary tree where each node represents a distinct tunnel (see right part of Figure 1). The root is the bounding tunnel, and the leaves are triangular tunnels that will not be split further. With each node n we associate the positive edge that represents the tunnel, $\epsilon(n)$, and with each leaf, we associate the negative triangle that fills this tunnel, $\tau(n)$. The tunnel-tree is ordered such that for any node n, the triangle of the rightmost leaf, $\tau(\text{TREE-MAX}(n))$, is the triangle that 'destroys' $\epsilon(n)$ and hence determines its persistence. A *void-tree* (or β_2-tree) of a 3D filter is defined in a similar fashion, only with positive triangles as nodes and negative tetrahedra as leaves.

A β_k-tree is constructed by running through the filter backwards as shown in Algorithm 1. Leaves are created when a negative $(k+1)$-simplex is encountered and the roots of leaves are connected when positive k-simplices are encountered.

Algorithm 1 Build a β_k-tree given a filter

Create a 'bounding node', n_b
for $i = n$ **to** 1 **do**
  if σ^i is a negative $(k+1)$-simplex **then**
    Create a new node, n, and set $\tau(n) \leftarrow \sigma^i$
  else if σ^i is a positive k-simplex **then**
    $(n_0, n_1) \leftarrow$ Nodes of the two $(k+1)$-simplices adjacent to σ^i
    $(n_0, n_1) \leftarrow (\text{ROOT}(n_0), \text{ROOT}(n_1))$
    Swap n_0 and n_1 if $\tau(\text{TREE-MAX}(n_0))$ is younger than $\tau(\text{TREE-MAX}(n_1))$
    Create a new node n with $n.left \leftarrow n_0$, $n.right \leftarrow n_1$, and $\epsilon(n.left) \leftarrow \sigma^i$
  end if
end for
return $\text{ROOT}(n_b)$

This is an abstract of a presentation given at CG:YRF 2012. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear in a conference with formal proceedings and/or in a journal.



+/-	+	+	+	+	+	+	+	-	-	-	-	-	-	+	+	+	-	+	+	-	-	-	+	-	-
σ^i	0	1	2	3	4	5	6	01	12	23	34	45	56	06	15	05	056	24	25	234	125	015	46	456	245
σ^j	∞	01	12	23	34	45	56							245	015	056		234	125				456		

Figure 1: **Left:** A 2D filter. For all positive k-simplices, σ^i, the $(k+1)$-simplex, σ^j, responsible for turning the k-cycle, represented by σ^i, into a k-boundary is indicated as well. **Right:** The tunnel-tree (β_1-tree) of the filter. Both $\epsilon(n)$ and $\tau(n)$ are shown for each node if they are defined.

In line 4, the $(k+1)$-simplex can be associated with its node using a hash-map. This ensures that locating the nodes of adjacent $(k+1)$-simplices in line 6 can be performed in constant time. In line 6, if one of the $(k+1)$-simplices adjacent to σ^i is not defined then the bounding node n_b is used instead. If σ^i has no adjacent $(k+1)$-simplices then a new node is created for n_0, and n_1 is set to n_b. Line 8 guarantees that the youngest simplex in a subtree can always be found by going to the far right in the tree using Tree-Max.

A β_k-tree may be arbitrarily unbalanced, so a straightforward implementation will run in $\mathcal{O}(n^2)$ time worst case. The Tree-Max-method can be improved to $\mathcal{O}(1)$ time by maintaining the maximum of each subtree as they are constructed. A data structure similar to disjoint-sets can be used to make the Root method run in $\mathcal{O}(\alpha(n))$-time, so the entire method runs in $\mathcal{O}(n\alpha(n))$ worst case time.

Applications

One attractive property of β_k-trees is that they give an alternative representation of the topological evolution of a filter. This can be used in several ways.

First, the fact that simplices in the subtree of a particular node will tend to be spatially close to each other gives rise to a new definition of *local persistence*. A particular edge, representing a tunnel, might be deemed particularly persistent if its subtree contains more than a certain number of nodes. Such a definition of persistence will not be affected by the addition of simplices outside the tunnel.

Using a Delaunay complex and the radius of the smallest empty circumcircle to generate an α-filter [3], the arrangement of a particular sub-tree also gives an indication of the shape of the corresponding feature. For instance, a node with an unbalanced sub-tree indicates a tunnel that is narrowing, whereas a balanced node indicates a constant width.

For some applications, a tree might be a better visualization of the topological evolution than e.g. k-triangles [2]. The above mentioned properties of locality can be computationally analyzed, but they can also be derived simply by inspecting β_k-trees. The length of edges in the tree can furthermore be scaled to reflect the difference in birth time of the $\epsilon(n)$ simplices.

Another interesting property of β_k-trees is that all $(k+1)$-simplices within a particular tunnel/void are easily identified by locating the node in the tree with the desired $\epsilon(n)$ and then collecting all leaves in the subtree using any tree-traversal method. In this manner the area of tunnels/volume of voids, for instance, is easily calculated.

Finally, any analysis method that works on trees is now applicable to topological evolutions. For instance the topology of two point-sets can be compared by finding the tree-edit-distance between the tunnel-trees (or void-trees) of their respective α-filters.

References

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